APPLICATION OF DIFFERENTIATION

CLASS XII (2025-26) SUJITHKUMAR KP

TYPE -1

1 Radius of a variable circle is changing at the rate of 5 cm/s. What is the radius of the circle at a time when its area is changing at the rate of 100 cm^2/s?

The area A of a circle with radius r is given by $A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \Rightarrow 100 = 2\pi r \times 5$

$\Rightarrow r = \frac{10}{\pi}$

2 Find the point on the curve $y = x^2$, where the rate of change of x-coordinate is equal to the rate of change of y-coordinate.

ANS: $y = x^2 \Rightarrow \frac{dy}{dt} = 2x\frac{dx}{dt}$

$\frac{dy}{dt} = \frac{dx}{dt}$

$\Rightarrow 1 = 2x \Rightarrow x = \frac{1}{2} \Rightarrow point\ is\ \left(\frac{1}{2}, \frac{1}{4}\right)$

3 The side of an equilateral triangle is increasing at the rate of 0.5 cm/s. Find the rate of increase of its perimeter

Let the side of the triangle be x

Then $\frac{da}{dt} = 0.5\ cm/s$

$P = 3a \Rightarrow \frac{dP}{dt} = 3\frac{da}{dt} = 3 \times 0.5 = 1.5\ cm/s$

4 If the rate of change of volume of a sphere is equal to the rate of change of its radius, then find the radius

Given $\frac{dV}{dt} = \frac{dr}{dt}$

$V = \frac{4}{3}\pi r^3 \Rightarrow 4\pi r^2 \times \frac{dr}{dt} = \frac{dr}{dt}$

$4\pi r^2 = 1 \Rightarrow r^2 = \frac{1}{4\pi} \Rightarrow r = \frac{1}{2\sqrt{\pi}}$ units

5 A stone is dropped into a quiet lake and waves move in circles at a speed of 5 cm per second. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

Let $r = radius\ of\ circular\ wave\ \ A = area.$

$\frac{dr}{dt} = 5\ cm/s$

$A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r\ \frac{dr}{dt} = 2\pi r \times 5 = 10\pi r$

$\frac{dA}{dt}\ at\ r = 8 = 10\pi \times 8 = 80\pi cm^2/s$

6 A balloon which always remains spherical has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x.

Diameter of the balloon $= \frac{3}{2}\,(2x+1)$

Radius of the balloon $= \frac{3}{4}\,(2x+1)$

$V = \frac{4}{3}\pi\left(\frac{3}{4}\,(2x+1)\right)^3 = \frac{9}{16}\,\pi\,(2x+1)^3$

$\frac{dV}{dt} = \frac{9}{16}\pi \times 3(2x+1)^2 = \frac{27}{8}\,\pi(2x+1)^2$

7 A spherical balloon is being inflated by pumping in 16 cm^3/s of gas. At the instant when balloon contains 36π cm^3 of gas, how fast is its radius increasing?

$\frac{dV}{dt} = 16\ cm^3/s$

$V = \frac{4}{3}\,\pi r^3 = 36\pi$

$\frac{dV}{dt} = 4\pi r^2\,\frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt}\ at\ r = 3 = \frac{16}{4\pi \times 9} = 0.14\ cm/s$

8 A particle moves along the curve 6y = *x*3 + 2. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the *x*-coordinate

$\frac{dy}{dt} = 8\,\frac{dx}{dt}$ Given curve is $6y = x^3 + 2$

$6\frac{dy}{dt} = 3x^2\,\frac{dx}{dt}$

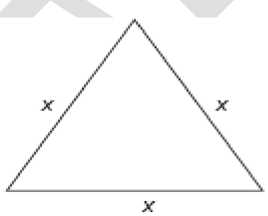
$6 \times 8 \times \frac{dx}{dt} = 3x^2\,\frac{dx}{dt} \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

Now substituting the value of x in the given equation of the curve, we get

$y = 11,\ -\frac{31}{3}$ Hence, points are (4, 11) and $\left(-4, -\frac{31}{3}\right)$

9 The side of an equilateral triangle is increasing at the rate of 2 cm/s. At what rate is its area increasing when the side of the triangle is 20 cm?

Let side of equilateral triangle be *x*.



$A = \frac{\sqrt{3}}{4}x^2$

$\frac{dA}{dt} = \frac{\sqrt{3}}{4} \times 2x \times \frac{dx}{dt} = \sqrt{3}x$

$\frac{dA}{dt}\ at\ x = 20 = 20\sqrt{3}cm^2/s$

$\frac{dx}{dt} = 2\ cm/sec$

10 The length *x* of a rectangle is decreasing at the rate of 5 cm/minute and the width *y* is increasing at the rate of 4 cm/minute. When *x* = 8 cm and y = 6 cm, find the rate of change of (*a*) the perimeter, and (*b*) the area of the rectangle

$\frac{dx}{dt} = -5\ cm/\min$, $\frac{dy}{dt} = 4\ cm/min$

(a) Perimeter of the rectangle $P = 2(x + y)$

$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-5 + 4) = -2\ cm/min$

(b) Area of rectangle A = xy

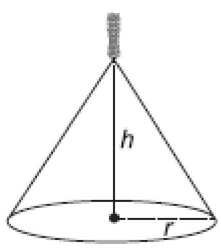
$\frac{dA}{dt} = x \times \frac{dy}{dt} + y \times \frac{dx}{dt} = 4x - 5y$

$\frac{dA}{dt}$ $at\ x = 8\ and\ y = 6 = \ 32 - 30 = 2\ cm^2/min$

11 A man 160 cm tall walks away from a source of light situated at the top of the pole 6m high, at the rate of 1.1 m/s. How fast is the length of his shadow increasing when he is 1m away from the pole?

(Ans: 0.4 m/s)

12 Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?
Let $h =$ be the height , V be the volume and r the radius of the base of the cone at time t.



$h = \frac{1}{6}r \Rightarrow r = 6h$

$\frac{dV}{dt} = 12cm^3/s$

V = $\frac{1}{3}\ \pi r^2 h\ \frac{1}{3}\pi(6h)^2 h = 12\pi h^3$

$\frac{dV}{dt} = 12\pi.3h^2 \times \frac{dh}{dt} \Rightarrow 12 = 36\ \pi h^2 \times \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{3\pi h^2}$

$\frac{dh}{dt}\ at\ h = 4is\ \ \frac{1}{3\pi \times 16} = \frac{1}{48\pi} cm/s$

13 A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?

Let foot of the ladder be x m away from the wall and y is height at time t.

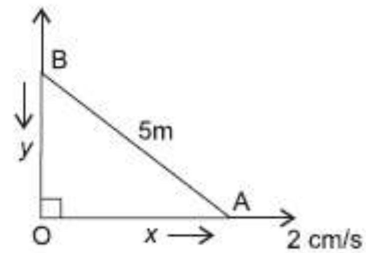
$$\therefore \quad \frac{dx}{dt} = 2 \text{ cm/s} \qquad \ldots(i)$$

We have $x^2 + y^2 = 25 \Rightarrow y = \sqrt{25 - x^2}$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{1}{2\sqrt{25-x^2}}(-2x)\frac{dx}{dt} = \frac{-2x}{\sqrt{25-x^2}} \text{ [from (i)]}$$

$$\left.\frac{dy}{dt}\right]_{x=4} = \frac{-8}{\sqrt{25-16}} = -\frac{8}{3} \text{ cm/s}$$

Hence, Height is decreasing at the rate of $\frac{8}{3}$ cm/s.



14 At what point of the ellipse $16x^2 + 9y^2 = 400$, does the ordinate decrease at the same rate at which the abscissa increases?

Let the point be (x, y) then $\frac{dy}{dt} = -\frac{dx}{dt}$

Differentiating both sides of the ellipse $16x^2 + 9y^2 = 400$ w.r.t. t, we get

$$16.2x.\frac{dx}{dt} + 18y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{16x}{9y}.\frac{dx}{dt} \Rightarrow 16x = 9y$$

Substituting in curve, we get

$$16\,x^2 + \frac{256}{9}\,x^2 = 400 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3 \text{ then } y = \pm\frac{16}{3}$$

Points are $\left(3, \frac{16}{3}\right)$ *and* $\left(-3, -\frac{16}{3}\right)$

15 An edge of a variable cube is increasing at the rate of 5cm per second. How fast is the volume increasing when the side is 15cm?

ANS: let x be the edge V be the volume.

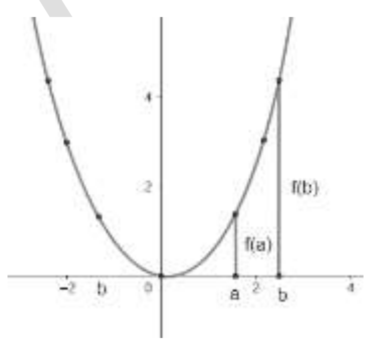
$\frac{dx}{dt} = 5\ cm/s,\ \ x = 15\ cm \qquad V = x^3$

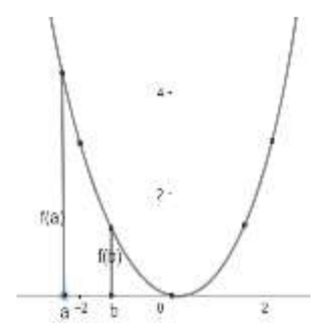
$$\frac{dV}{dt} = 3 \times 15^2 = 3375\ cm^3/sec$$

TYPE -2

1 Define increasing and decreasing functions.

ANS: A function is said to be an increasing function if the value of y increases with the increase in x.
A function is said to be a decreasing function if the value of y decreases with the increase in x.





As we move from left to right,
the height of the graph decreases

As we move from left to right,
the height of the graph increases

2 Show that the function given *by* $f(x) = 7x - 3$ is increasing on **R**.

ANS:

Let x_1 and x_2 be any two numbers in **R**. Then

$x_1 < x_2 \Rightarrow 7x_1 < 7x_2$

$7x_1 - 3 < 7x_2 - 3$

$f(x_1) < f(x_2)$

it follows that *f* is strictly increasing on **R**.

3 Find the intervals in which the function *f* given by $f(x) = x^2 - 4x + 6$ is (a) increasing (b) decreasing

ANS: $f(x) = x^2 - 4x + 6$

$f'(x) = 2x - 4, \quad f'(x) \Rightarrow 0$

$$2x - 4 = 0 \Rightarrow x = 2$$

gives $x = 2$. Now the point $x = 2$ divides the real line into two disjoint intervals namely $(-\infty, 2)$ and $(2, \infty)$

In $(-\infty, 2)$, $2x - 4 < 0$, f is decreasing in this interval

in $(2, \infty)$, $2x - 4 > 0$, *f* is increasing in this interval.

4 Find the intervals in which the function *f* given by $f(x) = 4x^3 - 6x^2 - 72x + 30$ is (a) increasing (b) decreasing.

ANS: $f(x) = 4x^3 - 6x^2 - 72x + 30$

$f'(x) = 12x^2 - 12x - 72$

$= 12(x^2 - x - 6)$

$= 12(x - 3)(x + 2)$

$f'(x) = 0 \Rightarrow x = 3, x = -2$

The points $x = -2$ and $x = 3$ divides the real line into three disjoint intervals, namely, $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$.

Interval	Sign of $f'(x)$	Nature of function
$(-\infty, -2)$	+	*f is increasing*
$(-2, 3)$	−	*f is decreasing*
$(3, \infty)$	+	*f is increasing*

$f'(x) = 12(x - 3)(x + 2)$

The function *f* is increasing in the intervals $(-\infty, -2)$, and $(3, \infty)$.
while the function is decreasing in the interval $(-2, 3)$.

5 Prove that the function $f(x) = x^3 - 3x^2 + 3x + 107$ is increasing in *R*.

ANS: $f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1)$

$= 3(x - 1)^2 >$ 0. Hence, function is increasing in *R*.

6 Show that the following functions are strictly increasing $f(x) = x^3 - 3x^2 + 4x$

ANS: $f(x) = x^3 - 3x^2 + 4x$

$f'(x) = 3x^2 - 6x + 4$

$= 3(x^2 - 2x + 1) + 1 = 3(x - 1)^2 + 1 > 0$

7 Show that the function $f(x) = x^2 - 5x + 1$ is neither increasing nor decreasing in [0, 5].

ANS: $f(x) = x^2 - 5x + 1$, $f'(x) = 2x - 5$

$2x - 5 = 0 \Rightarrow x = \frac{5}{2}$

For $\left(0, \frac{5}{2}\right)$, $f'(x) < 0$

$f(x) = x^2 - 5x + 1$ *is decreasing.*

For $\left(\frac{5}{2}, 5\right)$, $f'(x) > 0$

$f(x) = x^2 - 5x + 1$ is increasing.

8 TRY YOURSELF

1. Find the intervals in which the function f given by

$f(x) = 2x^2 - 3x$ is (a) increasing (b) decreasing

2. Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is (a) increasing (b) decreasing

3. Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor decreasing on $(-1, 1)$.

4) Find the intervals in which the function f given by (a) increasing (b) decreasing.

i) $f(x) = x^3 - 3x^2 + 3x - 100$

ii) $f(x) = 4x^3 - 6x^2 + 3x + 12$

iii) $f(x) = x^3 - 6x^2 + 12x - 16$

9 Find the maximum and minimum values if any of the function given by $f(x) = -(x-1)^2 + 10$.

$(x-1)^2 \geq 0$

$-(x-1)^2 \leq 0$ for $x \in R$

$-(x-1)^2 + 10 \leq 10$

$f(x) \leq 10$,

Maximum value = 10.

Minimum value = nil.

10 Find the maximum and minimum values if any of the function given by $f(x) = \sin 2x + 5$.

$-1 \leq \sin 2x \leq 1$

$-1 + 5 \leq \sin 2x + 5 \leq 1 + 5$

$4 \leq \sin 2x + 5 \leq 6$.

Maximum value = 6,

Minimum value = 4.

11 Find the maximum and minimum values, if any, of the function given by $f(x) = |\sin 4x + 3|$

$f(x) = |\sin 4x + 3|$

$-1 \leq \sin 4x \leq 1$

$2 \leq \sin 4x + 3 \leq 4$

$2 \leq |\sin 4x + 3| \leq 4$.

Minimum value = 2, Maximum value = 4.

12 Find the maximum and minimum value of the function $y = |x - 3| + 7, x \in R$

$|x - 3| \geq 0$ $\quad |x - 3| + 7 \geq 7$

$y \geq 7$

minimum value = 7, no maximum value

13 Find the least value of 'a' such that the function $f(x) = x^2 + ax + 1$ is strictly increasing on [1, 2].

ANS :

$$f(x) = x^2 + ax + 1$$

Differentiating both sides w.r.t. x, we get

$f'(x) = 2x + a$,

$1 < x < 2$. $\quad 2 < 2x < 4$

$2 + a < 2x + a < 4 + a$

or $4 + a > 2x + a > 2 + a$

For increasing $f'(x) > 0$

for least value $2 + a = 0$ $\quad a = -2$

14 Find the intervals in which the function f given by $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$ is (*i*) strictly increasing (*ii*) strictly decreasing.

$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$

$f'(x) = \frac{6}{5}x^3 - \frac{12}{5}x^2 - 6x + \frac{36}{5}$

$= \frac{6}{5}(x^3 - 2x^2 - 5x + 6)$

$\frac{6}{5}(x-1)(x-3)(x+2)$

$f'(x) = 0 \Rightarrow x = 1,\ x = 3,\ x = -2$

	$(-\infty,-2)$	$(-2,1)$	$(1,3)$	$(3,\infty)$
$(x-1)$	−	−	+	+
$(x-3)$	−	−	−	+
$(x+2)$	−	+	+	+
sign of $f'(x)$	−	+	−	+

Increasing in $(-2,1) \cup (3,\infty)$

decreasing in $(-\infty,-2) \cup (1,3)$

15 Find the intervals in which the function f given by $f(x) = 8 + 36x + 3x^2 - 2x^3$ is increasing or decreasing.

$f'(x) = -6x^2 + 6x + 36$

$= -6(x^2 - x - 6)$

$= -6(x-3)(x+2)$ ...

$f'(x) = 0$, $x = -2, 3$

	$x < -2$	$-2 < x < 3$	$x > 3$
-6	−	−	−
$x-3$	−	−	+
$x+2$	−	+	+
sign of $f'(x)$	−	+	−
Increasing/ decreasing	↓	↑	↓

Function increasing for (– 2, 3);
decreasing for
(– ∞, – 2) ∪ (3, ∞).

16 Find the intervals in which the function f given by $f(x) = \sin 3x, x \in \left[0, \frac{\pi}{2}\right]$ is (*i*) increasing (*ii*) decreasing.

ANS: Increasing for $\left(0, \frac{\pi}{6}\right)$ and decreasing for $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$

17 It is given that at $x = 1$, the function $f(x) = x^4 - 62x^2 + ax + 9$ attains its maximum value on the interval [0, 2]. Find the value of a.

$f'(x) = 4x^3 - 124x + a$, for a point of maximum

$f'(1) = 0$

$4 - 124 + a = 0$

$a = 120$

18 Show that the function $f(x) = \log|\cos x|$ is strictly decreasing in $\left(0, \frac{\pi}{2}\right)$.

$f'(x) = -\tan x$,

$\tan x > 0$ for $\left(0, \frac{\pi}{2}\right)$.

$f'(x) = -\tan x < 0$

Hence, function is strictly decreasing

19 Find the intervals in which the function *f* given by $f(x) = x - \sin x$ in [0, 2π] is increasing or decreasing

ANS: $f'(x) = 1 - \cos x = 2\sin^2\frac{x}{2}$, always positive, increasing in [0, 2π].

20 Show that the function $\tan^{-1}(\cos x + \sin x)$ is strictly increasing on $\left(0, \frac{\pi}{4}\right)$.

ANS: $y = \tan^{-1}(\cos x + \sin x)$

ALPHA

$$\frac{dy}{dx} = \frac{1}{1 + (cosx + sinx)^2}(-sinx + cosx)$$

$= \frac{-sinx+cosx}{1+(cosx+sinx)^2}$

$$1 + (cosx + sinx)^2 > 0$$

$\frac{dy}{dx} = \frac{-sinx+cosx}{1+(cosx+sinx)^2} > 0 \Rightarrow cosx - sinx > 0$

$cosx > sinx$

$$\Rightarrow x \in \left(0, \frac{\pi}{4}\right).$$

21 Show that the function f(x) = log $sinx$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$

Ans : f(x) = log $sinx$

$f'(\text{x}) = \cot \text{x}$

x $\in \left(0, \frac{\pi}{2}\right)$, cot x > 0, $f'(\text{x}) > 0$

f(x) is increasing on $\left(0, \frac{\pi}{2}\right)$

x $\in \left(\frac{\pi}{2}, \pi\right)$, cot x < 0, $f'(\text{x}) < 0$

f(x) is decreasing on $\left(\frac{\pi}{2}, \pi\right)$

22 Which of the following functions are strictly decreasing on $\left(0, \frac{\pi}{2}\right)$

i) $cosx$ ii) $cos2x$ iii) $cos3x$ iv) $tanx$

ANS: i) $y = cosx \Rightarrow \frac{dy}{dx} = -sinx < 0$, $\Rightarrow$ strictly decreasing

ii) $y = cos2x \Rightarrow \frac{dy}{dx} = -2sin2x$

x $\in \left(0, \frac{\pi}{2}\right) \Rightarrow 0 < x < \frac{\pi}{2}$

$0 < 2x < \pi$

$sin2x > 0 \Rightarrow$ $-2sin2x < 0 \Rightarrow$
strictly decreasing

iii) $y = cos3x$

$\Rightarrow \frac{dy}{dx} = -3\ sin3x$

x $\in \left(0, \frac{\pi}{2}\right) \Rightarrow 0 < x < \frac{\pi}{2}$, $0 < 3x < \frac{3\pi}{2}$

$3x \in (0, \pi)$ then $sin3x > 0$

$-3\ sin3x < 0 \Rightarrow$ f(x) is decreasing on $\left(0, \frac{\pi}{3}\right)$

$3x \in \left(\pi, \frac{3\pi}{2}\right)$ then $sin3x < 0$

$-3\ sin3x > 0$ f(x) is increasing.

23 HOME WORK

1.Find the intervals in which the function f given by

$f(x) = x^3 - x^2 - 1$ is increasing or decreasing.

ANS: increasing on $(-\infty, 0) \cup (4, \infty)$ decreasing on $(0, 4)$

2.Find the intervals in which the function f given by $f(x) = -2x^3 - 9x^2 - 12x + 1$ is increasing or decreasing

ANS: increasing on $(-2,-1)$ decreasing on $(-\infty, -2) \cup (-1, \infty)$

3. Find the intervals in which the function

$f(x) = \frac{x}{2} + \frac{2}{x}$, $x \neq 0$ is strictly increasing or decreasing

ANS: increasing on $(-\infty, -2) \cup (2, \infty)$ decreasing on $(-2, 2)$

4. Find the intervals in which the function

$f(x) = x^4 - 2x^2$ is strictly increasing or decreasing.

ANS: increasing on $(-1, 0) \cup (1, \infty)$ decreasing on $(-\infty, -1) \cup (0,1)$

5.Find the intervals in which the function *f* given by $f(x) = 2x^3 - 9x^2 + 12x + 15$ is strictly increasing or strictly decreasing.

ANS:1 Strictly increasing for $(-\infty, 1) \cup (2, \infty)$, strictly decreasing in $(1, 2)$

6. Prove that the function $f(x) = x^3 - 3x^2 + 3x + 107$ is increasing in *R*.

ANS:

$f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1)$

$= 3(x - 1)^2 > 0.$ Hence, function is increasing in *R*.

24 Find the local maximum and local minimum values of the function $f(x) = sinx + \frac{1}{2}cos2x,\ 0 < x < \frac{\pi}{2}$

$f(x) = sinx + \frac{1}{2}cos2x,\ 0 < x < \frac{\pi}{2}$

$f'(x) = cosx + \frac{1}{2}(-2sin2x) = cosx - sin2x$

$f'(x) = 0$ then

$cosx - sin2x = 0$

$cosx(1 - 2sinx) = 0$, $cosx = 0$, $sinx = \frac{1}{2}$ $x = \frac{\pi}{2}, \frac{\pi}{6}$

$f''(x) = -sinx - 2\,cos2x$

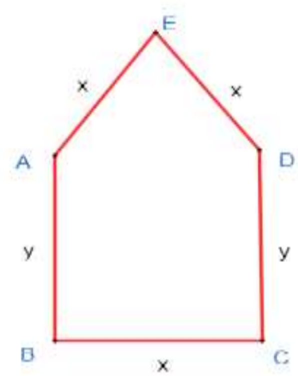
$f''\left(\frac{\pi}{6}\right) = -\frac{3}{2} < 0$

Local max. value = $\frac{3}{4}$ at $\frac{\pi}{6}$

$f''\left(\frac{\pi}{2}\right) = 1 > 0$, Local min. value =1/2 at $\frac{\pi}{2}$

25 A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, We need to find the dimensions of the rectangle that will produce the largest area of the window using second derivative test.

Let x m be the side of the equilateral triangle, and y m. be the length of the rectangle and Area of the window is A sq. m.



Let x m be the side of the equilateral triangle, and y m. be the length of the rectangle.

Perimeter of the window $= 3x + 2y = 12$

$\Rightarrow y = \frac{12-3x}{2}$

Area of the window , A = $xy + \frac{\sqrt{3}x^2}{4}$

$(6 - 3x) + \frac{\sqrt{3}\,x}{2} = 0$

$12 - 6x + \sqrt{3}\,x = 0$

$x = \frac{12}{6-\sqrt{3}} \frac{dA}{dx} = \frac{1}{2}(12 - 6x) + \frac{\sqrt{3}\,x}{2}$

$\frac{d^2A}{dx^2} = -3 + \frac{\sqrt{3}}{2} < 0$

A = $x\left(\frac{12-3x}{2}\right)+\frac{\sqrt{3}x^2}{4} = \left(\frac{12x-3x^2}{2}\right)+\frac{\sqrt{3}x^2}{4}$

$\frac{dA}{dx}=\frac{1}{2}\,(12-6x)+\frac{\sqrt{3}x}{2}$

$\frac{dA}{dx}=0 \Rightarrow \frac{1}{2}\,(12-6x)+\frac{\sqrt{3}\,x}{2}=0$

$A\ is\ maximum\ when\ \ x=\frac{12}{6-\sqrt{3}}$

$y=\frac{12-3x}{2}$ ie . $2y=12-3x$ = 12 - $\frac{36}{6-\sqrt{3}} = \frac{72-12\sqrt{3}-36}{6-\sqrt{3}}$

$y=\frac{18-6\sqrt{3}}{6-\sqrt{3}}$, $x=\frac{12}{6-\sqrt{3}}$

27 Given the graph of the function:

$f(x) = x^3 - 6x^2 + 9x + 15.$

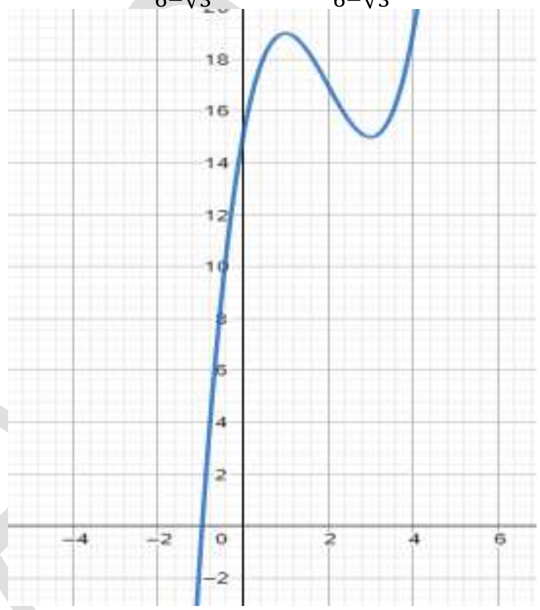
i) Find the critical point of the function.

ii) Find all the points of local maxima and local minima of the function.

iii) Find local minimum or local maximum values.

OR

Find the intervals in which the function is strictly increasing/strictly decreasing.



ANS: i) $f'(x)=3x^2-12x+9$

$f'(x)=3(x^2-4x+3)=0\ (x-1)(x-3)=0$

$x=1,\ x=3$

ii) $f''(x) = 6x-12$

$f''(1) = 6-12=-6<0$, $x=1$ *is a local maximum.*

$f''(3) = 6\times 3-12=6>0$, $x=3$ *is a local minimum*

iii) *local maximum* value is $f(1)=1^3-6\times 1^2+9+15=19$

local minimum value is $f(3)=3^3-6\times 3^2+9\times 3+15=15$

OR

$f'(x)=3(x^2-4x+3)\,)=0\ (x-1)(x-3)=0$

$x=1,\ x=3$

Intervals are (– ∞ ,1) (1, 3) , (3, ∞)

Interval	Sign of $f'(x)$	
(– ∞ ,1)	$+ve$	Increasing
(1, 3)	$-ve$	decreasing
(3, ∞)	$+ve$	Increasing

Increasing in (– ∞ ,1) ∪ (3, ∞) , decreasing in (1, 3)

28 The radius r cm of a blot of ink is increasing at the rate of 1.5 mm/sec. Find the rate at which the area A is increasing after 4 sec. (Ans: 0.18 $\pi cm^2/sec$)

29 Find the intervals in which the function f given by $f(x) = 8 + 36x + 3x^2 - 2x^3$ is increasing or decreasing.

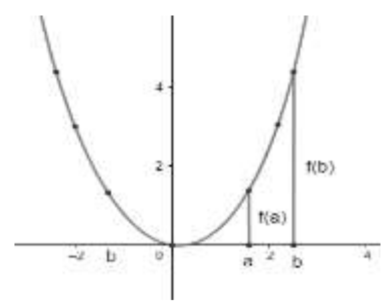
$f'(x) = -6x^2 + 6x + 36$
$= -6(x^2 - x - 6)$
$= -6(x-3)(x+2)$ ...
$f'(x) = 0$, $x = -2, 3$
Function increasing for $(-2, 3)$; decreasing for
$(-\infty, -2) \cup (3, \infty)$.

	$x < -2$	$-2 < x < 3$	$x > 3$
-6	$-$	$-$	$-$
$x - 3$	$-$	$-$	$+$
$x + 2$	$-$	$+$	$+$
sign of $f'(x)$	$-$	$+$	$-$
Increasing/ decreasing	↓	↑	↓

30 Separate $\left(0, \frac{\pi}{2}\right)$ into subintervals in which the function $f(x) = sin3x$ is increasing or decreasing
ANS: $f(x) = sin3x$
$f'(x) = 3cos3x$
$f'(x) > 0,\ cos3x > 0$
$0 < x < \frac{\pi}{2} \Rightarrow 0 < 3x < \frac{3\pi}{2}$
$f(x) = sin3x$ is increasing on
$0 < 3x < \frac{\pi}{2}$, ie $0 < x < \frac{\pi}{6}$
$f(x) = sin3x$ decreasing on
$\frac{\pi}{2} < 3x < \frac{3\pi}{2}$
ie $\frac{\pi}{6} < x < \frac{\pi}{2}$

31 Find all points of local maxima and local minima of the function *f* given by $y = x^2$. Find also local minimum or local maximum values.

ANS: *<u>First derivative Test</u>*
$y = f(x) = x^2$
$\frac{dy}{dx} = f'(x) = 2x$,
$\frac{dy}{dx} = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$, critical value is $x = 0$
For x slightly less than 0 , say $-\frac{1}{2}$
$f'\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right) = -1 < 0$
For x slightly greater than 0 , say 1
$f'(1) = 2(1) = 2 > 0$
$f'(x)$ changes sign from $-ve\ to + ve\ \ as$
$x\ increases$ through 0.
$x = 0$ is a local minimum and corresponding minimum value is $f(0) = 0^2 = 0$. *Second derivative Test,*

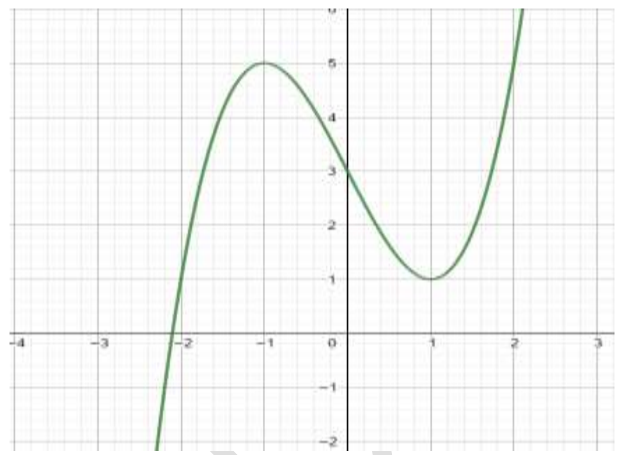


$y = f(x) = x^2$
$\frac{dy}{dx} = f'(x) = 2x,$
$f''(x) = 2 > 0$ so , $x = 0$ is a local minimum and corresponding minimum value is $f(0) = 0$.

32 Using First derivative Test, find all points of local maxima and local minima of the function *f* given by $f(x) = x^3 - 3x + 3$. Find also local minimum or local maximum values
$f(x) = x^3 - 3x + 3$.

$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$
$f'(x) = 0 \Rightarrow \quad x = 1$ and $x = -1$.
Thus, $x = \pm 1$ are the only critical points ,
for values close to 1 and to the left of 1, (say x = 0)
$f'(x) < 0$.

for values close to 1 and to the right of 1, (say x = 2)
$f'(x) > 0$ and
Therefore, by first derivative test, $x = 1$ is a point of local minima and local minimum value is
$f(1) = 1^3 - 3 + 3 = 1$
$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$
For $x = -1$,
$f'(x) > 0$, for values close to and to the left of -1 (say -2)

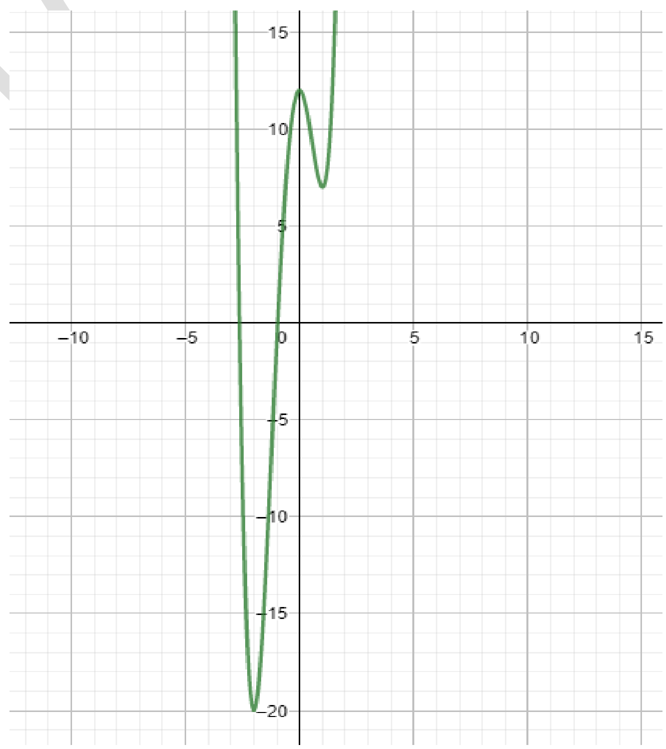


$f(x) = x^3 - 3x + 3$

and $f'(x) < 0$, for values close to and to the right of -1 (say 0)
Therefore, by first derivative test, $x = -1$ is a point of local maxima.
local maximum value is $f(-1) = 5$

33 Using second derivative test, find local maximum and local minimum values of the function f given by $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$.

ANS: $f'(x) = 12x^3 + 12x^2 - 24x = 12x(x - 1)(x + 2)$
or $f'(x) = 0$
$at \quad x = 0, \quad x = 1 \; and \quad x = -2.$
Now $f''(x) = 36x^2 + 24x - 24 = 12(3x^2 + 2x - 2)$
$f''(0) = -2 < 0$
$f''(1) = 12(3 \times 1^2 + 2 \times 1 - 2) = 36 > 0$
$f''(-2) = 12[3 \times (-2)^2 + 2 \times (-2) - 2] = 72 > 0$
by second derivative test, $x = 0$ is a point of local maxima and
local maximum value of f at $x = 0$ is $f(0) = 12$
while $x = 1$ and $x = -2$ are the points of local minima and
local minimum values of f at $x = -1$ and -2 are $f(1) = 7$ and $f(-2) = -20$, respectively.



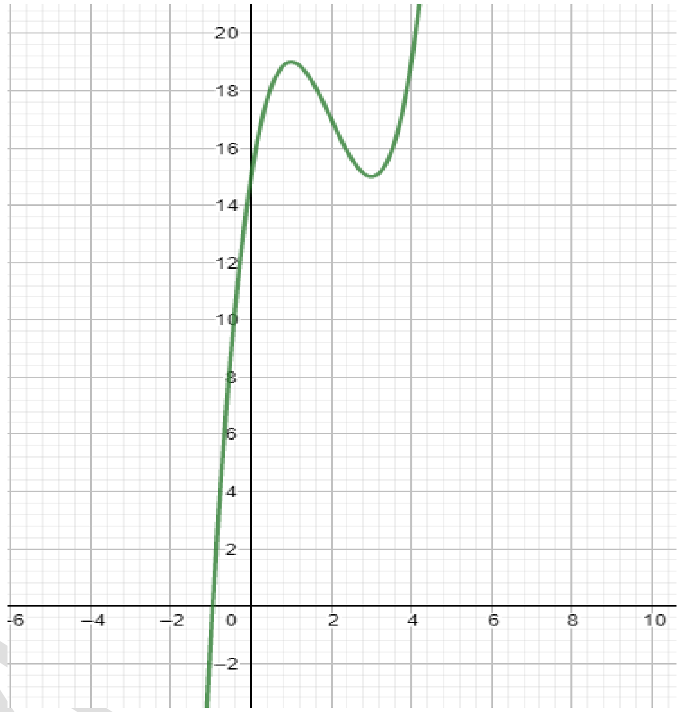
34 Find all the points of local maxima and local minima of the function f given by $f(x) = x^3 - 6x^2 + 9x + 15$. Find also local minimum or local maximum values.

ANS: $f'(x) = 3x^2 - 12x + 9$
$f'(x) = 3(x^2 - 4x + 3) = 0$ $(x-1)(x-3) = 0$ $x = 1,\ x = 3$
$f''(x) = 6x - 12$
$f''(1) = 6 - 12 = -6 < 0$
$f''(3) = 6 \times 3 - 12 = 6 > 0$
$f(x) = x^3 - 6x^2 + 9x + 15$

$x = 1$ *is a local maximum.*
local maximum value is
$f(1) = 1^3 - 6 \times 1^2 + 9 + 15 = 19$

$x = 3$ *is a local minimum*
local minimum value is
$f(3) = 3^3 - 6 \times 3^2 + 9 \times 3 + 15 = 15$

local maximum value = 19 at $x = 1$,
local minimum value = 15 at $x = 3$



35 Find the local minimum local minimum value $y = x^5 - 5x^4 + 5x^3 - 1$

$y = x^5 - 5x^4 + 5x^3 - 1$ $\frac{dy}{dx} = 5x^4 - 20x^3 + 15x^2$

$= 5x^2(x^2 - 4x + 3) = 5x^2(x-1)(x-3)$

$\frac{dy}{dx} = 0$, $5x^2(x^2 - 4x + 3) = 0$,

$x = 0,\ x = 1, x = 3$

$\frac{d^2y}{dx^2} = 20x^3 - 60x^2 + 30x$

$\frac{d^2y}{dx^2}$ at $x = 0$ is 0

Test fails ,

Go back to first derivative test.

$\frac{dy}{dx} = 5x^2(x-1)(x-3)$

$\frac{d^2y}{dx^2} = 20x^3 - 60x^2 + 30x$

When x slightly less than 0, $\frac{dy}{dx} > 0$

When x slightly greater than 0 , $\frac{dy}{dx} > 0$

$\frac{dy}{dx}$ does not change sign as x increases through 0

y is neither max. nor min. at x = 0.

$\frac{d^2y}{dx^2}$ at $x = 1$ is 20 – 60 + 30 = $-10 < 0$

$x = 1$ is a point of local max. and local max. value is

$y = 1^5 - 5 \times 1^4 + 5 \times 1^3 - 1 = 0$

$\frac{d^2y}{dx^2}$ at $x = 3$

is $20 \times 3^3 - 60 \times 3^2 + 30 \times 3 = 540 - 540 + 90 > 0$

At x = 3 point of local minimum.

local minimum value is

$$y = 3^5 - 5 \times 3^4 + 5 \times 3^3 - 1$$

$y =$ 243- 405 + 135- 1 = $-$ 28

36 HOME WORK

Find all the points of local maxima and local minima of the function f given below. Find also local minimum or local maximum values.

1. $x^3 - 3x$

ANS: $local\ maximum$ value = 2 at $x = -1$,

$local\ minimum$ value $= -$ 2 at $x = 1$

2. $(x-1)(x+2)^2$

ANS: $local\ maximum$ value = 2 at $x = -1$,

$local\ minimum$ value = -2 at $x = 1$

3. $\frac{x}{2} + \frac{2}{x}$, $x > 0$

ANS: $local\ minimum$ value = 2 at $x = 2$

(x = -2 discarded)

4. $\frac{1}{x^2+1}$,

ANS: $local\ maximum$ value = $\frac{1}{2}$ at $x = 0$.

5. $2x^3 - 21x^2 + 36x - 20$

ANS: $local\ maximum$ value = -3 at x =1

$local\ minimum$ value = -128 at $x = 6$

6. Find the local maxima and local minima of the cubic function f(x)= $x^3 - 3x^2 - 9x + 2$. Find also local minimum or local maximum values.

ANS : Max at x= -1 Max. value is 7

Min. at x = 2 Min. value is -20

7. Find the local extrema points of the function

f(x)= (x−a)e^x, where a is an arbitrary real number.

ANS: Local Min. $(a-1, -e^{a-1})$

37 Find all the points of local maxima and local minima, if any, of the function f given by

$f(x) = sin^4x + cos^4x$, $0 < x < \frac{\pi}{2}$. Find also local minimum or local maximum values.

ANS: $f(x) = sin^4x + cos^4x$, $0 < x < \frac{\pi}{2}$

$f'(x) = 4\ sin^3x\ cosx - 4\ cos^3x\ sinx$

$= 4sinx\ cosx(sin^2x - cos^2x)$

$= -2\ (2sinx.\ cosx)(cos^2x - sin^2x)$

$= -2sin2x.cos2x$

$f'(x) = = -sin4x$

$f''(x) = -4\ cos4x$

For local maximum or minimum , $f'(x) = 0$

$= -2\ (2sinx.\ cosx)(cos^2x - sin^2x)$ =0

$cos^2x - sin^2x$= 0 ⇒ tan^2x= 1 ⇒ $x = \frac{\pi}{4}$

ANS: $f(x) = sin^4x + cos^4x$, $0 < x < \frac{\pi}{2}$

$f'(x) = 4\ sin^3x\ cosx - 4\ cos^3x\ sinx$

$= 4sinx\ cosx(sin^2x - cos^2x)$

$= -2\ (2sinx.\ cosx)(cos^2x - sin^2x)$

$= -2sin2x.cos2x$

$f'(x) = = -sin4x$

$f''(x) = -4\ cos4x$

For local maximum or minimum , $f'(x) = 0$

$= -2\ (2sinx.\ cosx)(cos^2x - sin^2x)$ =0

$cos^2x - sin^2x$= 0 ⇒ tan^2x= 1 ⇒ $x = \frac{\pi}{4}$

38 Find the local maximum and the local minimum values, if any, for the function $f(x) = sin\,2x - x$, in $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Also indicate the points at which local maximum and local minimum exist.

ANS: $f(x) = sin\,2x - x$

$f'(x) = 2cos2x - 1$

$f'(x) = 0 \Rightarrow \quad 2cos2x - 1 = 0$

$\Rightarrow\ cos2x = \frac{1}{2}$

$\Rightarrow\ 2x = \pm\frac{\pi}{3}$

$\Rightarrow\ x = \pm\ \frac{\pi}{6}$

$f'(x) = 2cos2x - 1$

$f''(x) = -4\ sin2x$

at $x = \frac{\pi}{6}$, $f''\left(\frac{\pi}{6}\right) = -4\ sin\frac{\pi}{3} = -4 \times \frac{\sqrt{3}}{2} < 0$

$x = -\frac{\pi}{6}$, $f''\left(-\frac{\pi}{6}\right) = -4\ sin\frac{-\pi}{3} = 4 \times \frac{\sqrt{3}}{2} > 0$

local maximum at $x = \frac{\pi}{6}$,

maximum value = $f\left(\frac{\pi}{6}\right) = \sin\left(2 \times \frac{\pi}{6}\right) - \frac{\pi}{6}$

$= \frac{\sqrt{3}}{2} - \frac{\pi}{6}$

local minimum at $x = \frac{-\pi}{6}$

minimum value = $f\left(\frac{-\pi}{6}\right) = \sin\left(2 \times \frac{-\pi}{6}\right) + \frac{\pi}{6} = \frac{\pi}{6} - \frac{\sqrt{3}}{2}$

39 Find the absolute maximum value and the absolute minimum value for the function $(x) = sin\,x + cos\,x \ \ x \in [0, \pi]$.

ANS: $f'(x) = cos\,x - sin\,x$

For absolute maximum or minimum $f'(x) = 0$

$cos\,x - sin\,x = 0$

$\Rightarrow cos\,x = sin\,x \Rightarrow \ x = \frac{\pi}{4}$

$f(0) = sin\,0 + cos\,0 = 1$

$f(\pi) = sin\,\pi + cos\,\pi = -1$

$f\left(\frac{\pi}{4}\right) = sin\frac{\pi}{4} + cos\frac{\pi}{4} = \sqrt{2}$

absolute maximum value is $\sqrt{2}$ at x = $\frac{\pi}{4}$

absolute minimum value is -1 at x = π

40 Find the absolute maximum value and the absolute minimum value for the function

$f(x) = \frac{x+1}{\sqrt{x^2+1}}$, $0 \leq x \leq 2$

ANS: $f'(x) = \frac{\sqrt{x^2+1}\,.\,1 - (x+1)\frac{2x}{2\sqrt{x^2+1}}}{x^2+1}$

ALPHA12

$= \frac{x^2+1-x^2-x}{(x^2+1)^{\frac{3}{2}}} = \frac{1-x}{(x^2+1)^{\frac{3}{2}}}$

$f'(x) = 0 \Rightarrow \; x = 1$

Critical value is x = 1 which is $0 \le x \le 2$

$f(x) = \frac{x+1}{\sqrt{x^2+1}}$

$$f(0) = 1$$

$f(1) = \sqrt{2}$ (1.414)

$f(2) = \frac{3}{\sqrt{5}}$ = (1.34)

absolute maximum value is $\sqrt{2}$ at x =1

absolute minimum value is 1 at x = 0

41 At what points in the interval $[0, 2\pi]$ does the function $sin2x$ attain its maximum .

ANS: $f(x) = sin2x$

$f'(x) = 2\,cos2x$

$f'(x) = 0 \quad \Rightarrow \; 2\,cos2x = 0$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

critical values of $sin2x$ in $[0,2\pi]$ are

$\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$f(x) = sin2x,$ x = $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$f(0) = sin0 = 0$

$f\left(\frac{\pi}{4}\right) = \sin(2 \times \frac{\pi}{4}) = 1$

$f\left(\frac{3\pi}{4}\right) = sin2\left(\frac{3\pi}{4}\right) = -1$

$f\left(\frac{5\pi}{4}\right) = sin2\left(\frac{5\pi}{4}\right) = 1$

$f\left(\frac{7\pi}{4}\right) = sin2\left(\frac{7\pi}{4}\right) = -1$

$$f(\pi) = sin\pi = 0$$

At $\left(\frac{\pi}{4}, 1\right), \left(\frac{5\pi}{4}, 1\right)$ the function attains max. values

42 What is the maximum value of the function $sinx + cosx$.

ANS: $f(x) = sinx + cosx$.

$f'(x) = cosx - sinx$

$f'(x) = 0 \quad \Rightarrow cosx - sinx = 0 \quad x = \frac{\pi}{4}$

$f''(x) = -sinx - cosx$

$f''(\frac{\pi}{4}) = -sin\frac{\pi}{4} - cos\frac{\pi}{4} = -\sqrt{2} < 0$

$$f\left(\frac{\pi}{4}\right) = sin\frac{\pi}{4} + cos\frac{\pi}{4} = \sqrt{2}.$$

43 Prove that the following function do not have maxima or minima.

i) e^x ii) $logx$ iii) $x + 2$ iv) $x^3 + x^2 + x + 1$

ANS: i) y = e^x

$\frac{dy}{dx} = e^x$

$\frac{dy}{dx} = 0$, $e^x = 0$ which is not possible for any real x. so no critical value of e^x.

ii) y = $logx$

$\frac{dy}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = 0$, $\frac{1}{x}$ = 0 which is not possible for any real x.

45 HOME WORK

1) Find the maximum value of $2x^2 - 24x + 107$ in the interval [1,3].

ANS: 89 at x = 3

2) Find all points of local maxima and local minima , if any, of $f(x) = x^3 - 6x^2 + 9x + 7$ Also find the max. min. values .

ANS: 11 at x = 1, 7 at x = 3

3) Find all points of local maxima and local minima , if any, of the following function . Also find the max, min. values.

i) $f(x) = 2x^3 - 24x + 107$

ii) $f(x) = x^3 + 4x^2 - 3x + 1$

iii) $f(x) = 3x^3 - 4x + 2$

iv) $f(x) = 2x^3 - 3x^2 - 12x + 4$

4) Find all points of local maxima and local minima , if any, of the following function . Also find the max, min. values.

i) $y = \frac{x^4}{x-1}$, $x \neq 1$

ii) $y = sinx - cosx,\ 0 < x < 2\pi$

5 Find the local extrema of the function

f(x) = $x^4 - 8x^3 + 22x^2 - 24x + 1$

ANS: Minimum −8 at x= 1, x = 3

Maximum −7 at x = 2

46 Show that of all the rectangles of given area the square has the smallest perimeter.

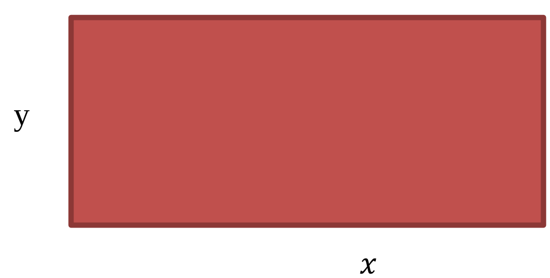
ANS: $A = xy$ (given), $P = 2(x + y) =$

$2\left(x + \frac{A}{x}\right)$

$\frac{dP}{dx} = 2\left(1 - \frac{A}{x^2}\right)$

$\frac{dP}{dx} = 0 \Rightarrow 2\left(1 - \frac{A}{x^2}\right) = 0$

$\Rightarrow A = x^2 \Rightarrow x = \sqrt{A}$



$\frac{d^2P}{dx^2} = \frac{4A}{x^3}$

$\frac{d^2P}{dx^2}$ at x $= \sqrt{A}$ >0

Hence, perimeter is minimum for x $= \sqrt{A}$

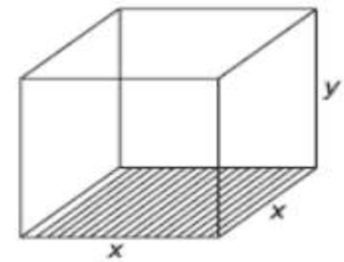
ie $x^2 = A = xy$

$\Rightarrow x = y$, rectangle is a square.

47 An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the material will be the least when the depth of the tank is half of its width.

ANS :

Let x be the side of the square base and y be height of the tank



$V = x2y$ ....(i) ; $S = 4xy + x2$ ....(ii)

$y = \frac{v}{x^2}$ sub. in (ii) $S = 4x . \frac{v}{x^2} + x2$

$S = \frac{4v}{x} + x2$ differentiate, $\frac{dS}{dx} = \frac{-4v}{x^2} + 2x$

$\frac{dS}{dx} = 0 \Rightarrow \frac{-4v}{x^2} + 2x = 0 \Rightarrow -4v + 2x^3 = 0$ ie

$2v = x^3$

$\frac{dS}{dx} = \frac{-4v}{x^2} + 2x$

$\frac{d^2s}{dx^2} = \frac{-4v(-2)}{x^3} + 2 = \frac{8v}{x^3} + 2$

$\frac{d^2s}{dx^2}$ at $x^3 = 2v$ is $\frac{8v}{2v} + 2 > 0$

surface area is minimum when $x^3 = 2v$

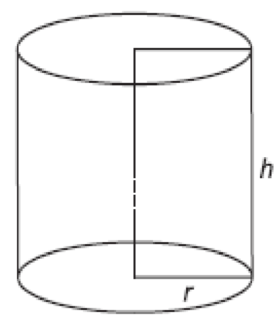
$\Rightarrow x^3 = 2x2y$

$x = 2y$

Or $y = \frac{x}{2}$

depth of the tank is half of its width.

48 Show that the height of a closed right circular cylinder of given surface and maximum volume is equal to the diameter of the base.



ANS:

Let *r* be radius of base and *h* be the height of a given cylinder

Given $S = 2\pi rh + 2\pi r2 \Rightarrow h = \frac{S - 2\pi r2}{2\pi r}$

Volume of cylinder $V = \pi r^2 h = \pi r^2 \left[\frac{S - 2\pi r2}{2\pi r}\right]$

$V = \frac{1}{2}[Sr - 2\pi r^3]$

$$\frac{dV}{dr} = \frac{1}{2}[S - 6\pi r^2]$$

$\frac{dV}{dr} = 0 \Rightarrow \frac{1}{2}[S - 6\pi r^2] = 0$

$\Rightarrow r = \sqrt{\frac{S}{6\pi}}$

$\frac{d^2V}{dr^2} = \frac{1}{2}(-12\pi r) = -6\pi r$

$\frac{d^2V}{dr^2}$ at $r = \sqrt{\frac{S}{6\pi}} = -6\pi \times \sqrt{\frac{S}{6\pi}} < 0$

ALPHA 12

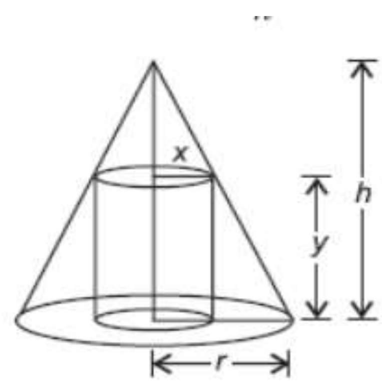
volume is Maximum for

$r = \sqrt{\frac{S}{6\pi}}$ $squaring, \quad r^2 = \frac{S}{6\pi} \Rightarrow 6\pi r^2 = S$

$6\pi r^2 = 2\pi rh + 2\pi r^2 \Rightarrow 2r^2 = rh \quad \Rightarrow \; h = 2r$

Hence, height of the cylinder is equal to the diameter of the base

49 Prove that, the radius of the right circular cylinder of greatest curved surface which can be inscribed in a given cone, is half of that of the cone.



ANS :

ANS: Let x be radius of the base and y be height of a cylinder enclosed in a given cone of base radius r and height h..

Curved surface of cylinder , $C = 2\pi xy$ ......(i)

$C = 2\pi xy$ ......(i)

$\frac{x}{r} = \frac{h-y}{h} \Rightarrow x = \frac{r}{h}(h-y)$ ---(ii)

$C = 2\pi . \frac{r}{h}(hy - y^2)$

$\frac{dC}{dy} = 2\pi . \frac{r}{h}(h - 2y), \quad \frac{dC}{dy} = 0 \Rightarrow 2\pi \frac{r}{h}(h-2y) = 0$

$\Rightarrow y = \frac{h}{2}$

$\Rightarrow \frac{d^2C}{dy^2} = \frac{2\pi r}{h} \times -2 < 0 \; for \; y = \frac{h}{2}$

$x = \frac{r}{h}\left(h - \frac{h}{2}\right) = \frac{r}{h} \times \frac{h}{2}$

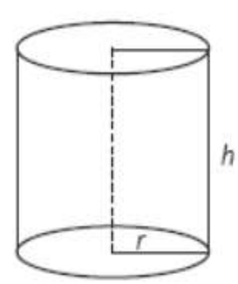
$x = \frac{r}{2}$

50 Of all the closed right circular cylindrical cans of volume $128\pi \; cm^3$, find the dimensions of the can which has the minimum surface area.

ANS :

Let r be radius of base and h be the height of a closed right circular cylinder of volume $128\pi \; cm^3$,

$$V = \pi r^2 h = 128\pi\_\_\_(i)$$



$\Rightarrow h = \frac{128\pi}{\pi r^2}$

$$S = 2\pi rh + 2\pi r^2 S = 2\pi r . \frac{128\pi}{\pi r^2} + 2\pi r^2 S$$

$$= \frac{256\pi}{r} + 2\pi r^2 \frac{dS}{dr} = \frac{-256\pi}{r^2} + 4\pi r$$

$$\frac{dS}{dr} = 0 \Rightarrow \frac{-256\pi}{r^2} + 4\pi r = 0$$

$\Rightarrow 4\pi r = \frac{256\pi}{r^2}$

$$r^2 = 64 \Rightarrow r = 4$$

$$\frac{dS}{dr} = \frac{-256\pi}{r^2} + 4\pi r$$

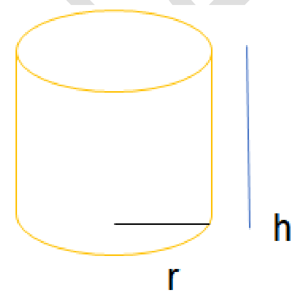
$$\frac{d^2S}{dr^2} = -\frac{512\pi}{r^3} + 4\pi \Rightarrow$$

$$\frac{d^2S}{dr^2} \; at \; r = 4 = -\frac{512\pi}{4^3} + 4\pi < 0$$

Hence, for $r = 4$, surface area is minimum
$128\pi = \pi(4)^2 h \Rightarrow h = 8$ [from (*i*)]   Hence, $r = 4$ cm and $h = 8$ cm, for minimum surface area.

51 Show that the right circular cylinder of given volume open at the top has minimum total surface area, provided its height is equal to radius of its base.

ANS: V= $\pi r^2 h \Rightarrow h = \frac{V}{\pi r^2}$



$S = 2\pi r h + \pi r^2$ (open)

$S = 2\pi r . \frac{V}{\pi r^2} + \pi r^2$

$S = \frac{2V}{r} + \pi r^2$

$\frac{dS}{dr} = \frac{-2V}{r^2} + 2\pi r$

$\frac{dS}{dr} = 0 \Rightarrow \quad \frac{-2V}{r^2} + 2\pi r = 0$

$2\pi r = \frac{2V}{r^2} \Rightarrow \; r^3 = V/\pi$

$$\frac{dS}{dr} = \frac{-2V}{r^2} + 2\pi r$$

$$\frac{d^2S}{dr^2} = \frac{4V}{r^3} + 2\pi$$

$\frac{d^2S}{dr^2} \; at \; r^3 = V/\pi = \frac{4V}{V/\pi} + 2\pi > 0$

Surface area is minimum when $r^3 = V/\pi$

V= $\pi r^2 h = \pi r^3$

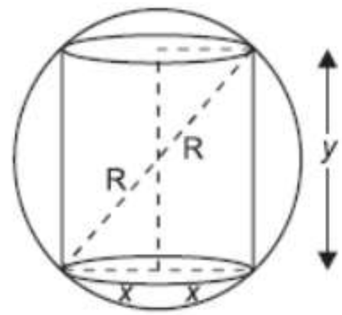
$$h = r$$

Height is equal to radius of its base.

Show that the semi-vertical angle of the cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$

52 Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

ANS: Let x be radius of base and y height of a cylinder which is inscribed in a sphere of radius R.



By Pythagoras theorem, $(2x)^2 + y^2 = (2r)^2$

$$4x^2 + y^2 = 4R^2 V = \pi x^2 y V = \pi y \left(\frac{4R^2 - y^2}{4}\right)$$

$$= \frac{\pi}{4}(4R^2 y - y^3)$$

$$\frac{dV}{dy} = \frac{\pi}{4}(4R^2 - 3y^2)$$

$\frac{dV}{dy} = 0 \Rightarrow \frac{\pi}{4}(4R^2 - 3y^2) = 0$

$3y^2 = 4\,R^2$

$$y = \frac{2R}{\sqrt{3}}$$

$$\frac{d^2V}{dy^2} = \frac{-3\pi y}{2} < 0 for\ y = \frac{2R}{\sqrt{3}}$$

Now putting the value of y in eq. (*ii*), we get Maximum volume $\frac{4\pi R^3}{3\sqrt{3}}$ cubic unit

53 A square piece of tin of side 18 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. What should be the side of the square to be cut off so that the volume of the box is maximum? Also find the maximum volume

ANS :Volume of the box, $V = (18 - 2x)^2 \,.\, x$,

where x is the side of the square cut off from each corner.
($l = 18 - 2x$, $b = 18 - 2x$, $h = x$)

$$V = (18 - 2x)2.\ x$$

$\frac{dV}{dx} = \left(18 - 2x\right)^2 + x\,.\,2(18 - 2x)(-2) = (18 - 2x)(18 - 6x)$

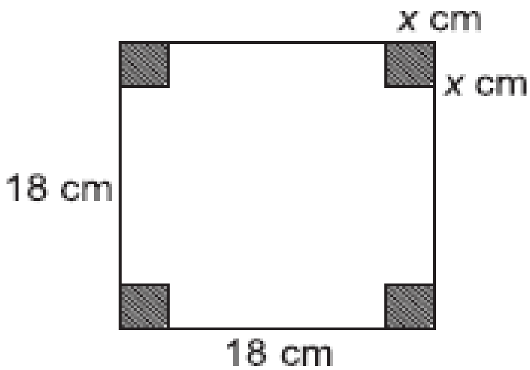
$\frac{dV}{dx} = 0 \Rightarrow \left(18 - 2x\right)\left(18 - 6x\right) = 0$

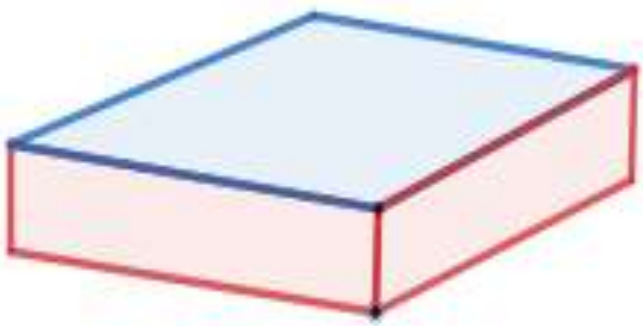
$\Rightarrow x = 9,\ x = 3$ ($x = 9$ discarded)

$\frac{d^2V}{dx^2} = (18 - 2x)(-6) + (18 - 6x)(-2)$

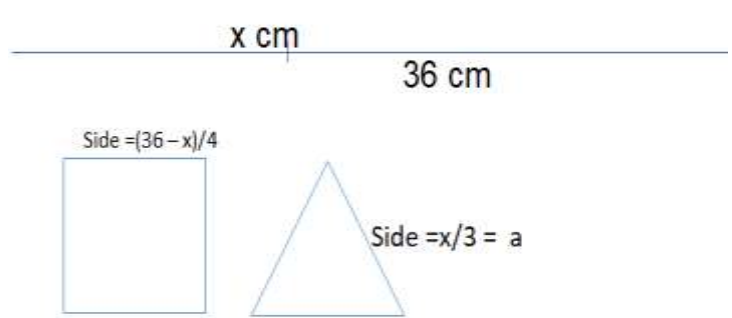
$\frac{d^2V}{dx^2}$ at x = 3 is $(18 - 6)(-6) + 0 = -72 < 0$. Hence, volume is maximum for $x = 3$

Maximum volume = $(18 - 6)^2 \times 3 = 432\ m^3$





54 A wire of length 36 cm is cut into two pieces. One of the pieces is turned in the form of a square and the other in the form of an equilateral triangle. Find the length of each piece so that the sum of the areas of the two be minimum.



55 A rectangle is inscribed in a semi- circle of radius r with one of its sides on the diameter of the semi- circle .find the dimensions of the rectangle, so that its area is maximum. Also find the maximum area.

Let x and y be the sides of the rectangle

$\left(\frac{x}{2}\right)^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - \frac{x^2}{4}$

Area of rectangle A = xy $\quad A^2 = x^2y^2$

Let Z = $A^2 = x^2y^2$

Z = $x^2(r^2 - \frac{x^2}{4})$

Z = $x^2r^2 - \frac{x^4}{4}$

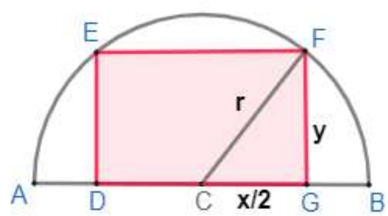
$\frac{dZ}{dx} = 2r^2x - x^3.$

$$\frac{dZ}{dx} = 0 \Rightarrow \quad 2r^2x - x^3 = 0$$

$\Rightarrow$ x$(2r^2 - x^2) = 0, \quad x \neq 0\,, x^2 = 2r^2 \Rightarrow x = \sqrt{2}\,r$

$$\frac{d^2Z}{dx^2}\ at\ x = \sqrt{2}\,r < 0$$

Maximum area = A = xy $= \sqrt{2}\,r\sqrt{\left(r^2 - \frac{x^2}{4}\right)} = r^2$



56 A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, find the dimensions of the rectangle that will produce the largest area of the window.

Let x m be the side of the equilateral triangle, and y m. be the length of the rectangle.

Perimeter of the window = $3x + 2y = 12$

$\Rightarrow y = \frac{12-3x}{2}$

Area of the window , A = $xy + \frac{\sqrt{3}\,x^2}{4}$

A = $x\left(\frac{12-3x}{2}\right) + \frac{\sqrt{3}\,x^2}{4} = \left(\frac{12x-3x^2}{2}\right) + \frac{\sqrt{3}\,x^2}{4}$

$\frac{dA}{dx} = \frac{1}{2}\,(12 - 6x) + \frac{\sqrt{3}\,x}{2}$

$\frac{dA}{dx} = 0 \Rightarrow \frac{1}{2}\,(12 - 6x) + \frac{\sqrt{3}\,x}{2} = 0$

$(6 - 3x) + \frac{\sqrt{3}\,x}{2} = 0$

$12 - 6x + \sqrt{3}\,x = 0$

$x = \frac{12}{6-\sqrt{3}} \frac{dA}{dx} = \frac{1}{2}\,(12 - 6x) + \frac{\sqrt{3}\,x}{2}$

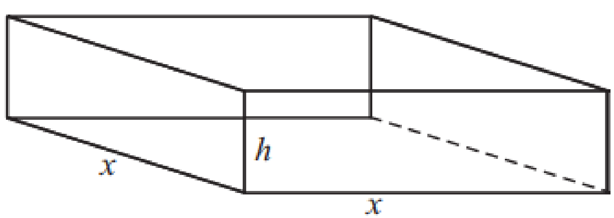
$\frac{d^2A}{dx^2} = -3 + \frac{\sqrt{3}}{2} < 0$

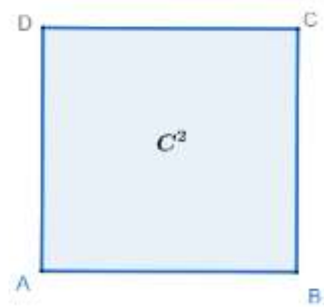
$A\ is\ maximum\ when\ \ x = \frac{12}{6-\sqrt{3}}$

$y = \frac{12-3x}{2}$ ie . $2y = 12 - 3x = 12 - \frac{36}{6-\sqrt{3}} = \frac{72-12\sqrt{3}-36}{6-\sqrt{3}}$

$y = \frac{18-6\sqrt{3}}{6-\sqrt{3}}$, $x = \frac{12}{6-\sqrt{3}}$

57 An open box with a square base is to be made of a given quantity of metal sheet of area c^2. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$.





ANS: Let the length, breadth and height of open box with square base be $x,\ x\ and\ h$ unit respectively. If V be the volume of box then $V = x.x.\ h$

$\Rightarrow \quad V = x^2h \quad ....(i)$

Also $\quad c^2 = x^2 + 4xh$

$\Rightarrow \quad h = \frac{c^2 - x^2}{4x}$

Putting it in (i) we get

$$V = \frac{x^2\,(c^2 - x^2)}{4x} \quad \Rightarrow \quad V = \frac{c^2 x}{4} - \frac{x^3}{4}$$

Differentiating w.r.t. x we get

$$\frac{dV}{dx} = \frac{c^2}{4} - \frac{3x^2}{4}$$

Now for maxima or minima

$$\frac{dV}{dx} = 0$$

$$\Rightarrow \qquad \frac{c^2}{4} - \frac{3x^2}{4} = 0 \qquad \Rightarrow \qquad \frac{3x^2}{4} = \frac{c^2}{4}$$

$$\Rightarrow \qquad x^2 = \frac{c^2}{3} \qquad \Rightarrow \qquad x = \frac{c}{\sqrt{3}}$$

Now, $\frac{d^2V}{dx^2} = -\frac{6x}{4} = -\frac{3x}{2}$

$$\therefore \qquad \left[\frac{d^2V}{dx^2}\right]_{x=c/\sqrt{3}} = -\frac{3c}{2\sqrt{3}} = -\text{ve}.$$

Hence, for $x = \frac{c}{\sqrt{3}}$ volume of box is maximum.

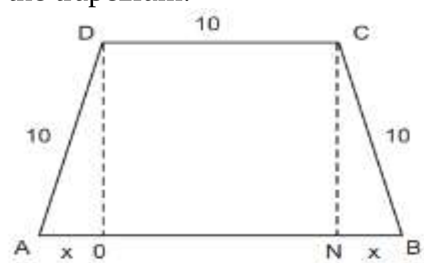
$$\therefore \qquad h = \frac{c^2 - x^2}{4x}$$

$$= \frac{c^2 - \frac{c^2}{3}}{4\frac{c}{\sqrt{3}}} = \frac{2c^2}{3} \times \frac{\sqrt{3}}{4c} = \frac{c}{2\sqrt{3}}$$

Therefore maximum volume $= x^2.h$

$$= \frac{c^2}{3} \cdot \frac{c}{2\sqrt{3}} = \frac{c^3}{6\sqrt{3}}$$

58 If the length of three sides of a trapezium other than the base are equal to 10 cm, then find the maximum area of the trapezium.



ANS:
Let DO = h
A = $\frac{1}{2}\, h(a+b)$
$h^2 = 10^2 - x^2$
$AD = DC = BC = 10$ cm.
Let $AO = NB = x$ cm.

$$\text{Area } (A) = \frac{1}{2}(AB + DC).DO$$

$$= \frac{1}{2}(10 + 2x + 10)\sqrt{100 - x^2}$$

$$\therefore \qquad A = (x + 10)\sqrt{100 - x^2}$$

Differentiating w.r.t. x, we get

$$\frac{dA}{dx} = (x+10) . \frac{1}{2\sqrt{100 - x^2}}(-2x) + \sqrt{100 - x^2}\,.1$$

$$= \frac{-x(x+10) + (100 - x^2)}{\sqrt{100 - x^2}} = \frac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}}$$

For maximum area, $\frac{dA}{dx} = 0$

$\Rightarrow \quad 2x^2 + 10x - 100 = 0$ or $x^2 + 5x - 50 = 0$

$\Rightarrow \quad (x+10)(x-5) = 0 \quad \Rightarrow \quad x = 5, -10$

$\Rightarrow \quad x = 5$

Now again differentiating w.r.t. x, we get

$$\frac{d^2A}{dx^2} = \frac{\sqrt{100-x^2}\,(-4x-10) - (-2x^2 - 10x + 100)\,.\,\frac{(-2x)}{2\sqrt{100-x^2}}}{(100-x^2)}$$

For $x = 5$

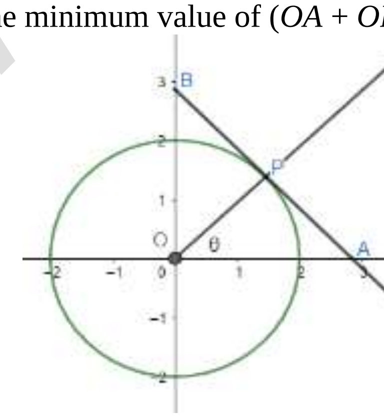
$$\frac{d^2A}{dx^2} = \frac{\sqrt{100-25}\,(-20-10) - 0}{(100-25)} = \frac{\sqrt{75}(-30)}{75} < 0$$

$\therefore$ For $x = 5$, area is maximum

$$A_{max} = (5+10)\sqrt{100-25}\ \text{cm}^2 \qquad \text{[Using equation } (i)]$$

$$= 15\sqrt{75}\ \text{cm}^2 = 75\sqrt{3}\ \text{cm}^2$$

59 Tangent to the circle $x^2 + y^2 = 4$ at any point on it in the first quadrant makes intercepts OA and OB on x and y-axes respectively, O being the centre of the circle. Find the minimum value of $(OA + OB)$.



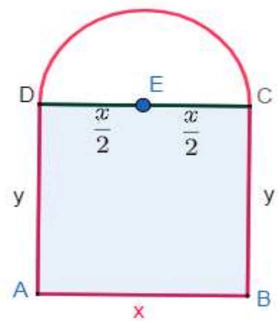
60 Show that the rectangle of maximum area that can be inscribed in a circle is a square.

61 Show that the altitude of a right circular cone of maximum volume that can be inscribed in a sphere of radius R is $\frac{4R}{3}$

62 A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 metres. Find the dimensions of the window so as to admit maximum light through the whole opening.

ANS: $\frac{20}{\pi+4}, \frac{10}{\pi+4}$

ANS:Let x and y be the length and width of rectangle part of window respectively. Let A be the opening area of window which admit Light. Obviously, for admitting the maximum light through the opening, A must be maximum.



A = Area of rectangle + Area of semi-circle

63 HOME WORK

1. A wire of length 28 metres is to be cut into two pieces. One of the pieces is to be made into a circle and the other into a square. What should be the length of the two pieces so that the combined area of the square and the circle is minimum? ANS: $\frac{28\pi}{4+\pi}, \frac{112}{4+\pi}$

2. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle 30° is $\frac{4\pi}{81}h^3$.

3. Find the volume of the largest cylinder that can be inscribed in a sphere of radius r.

4. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3 . If building of tank costs Rs. 70 per sq. metre for the base and Rs. 45 per sq. metre for sides, what is the cost of least expensive tank?
ANS: Rs. 1000

5. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 tan^2\alpha$.

6. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius R is $\frac{4R}{3}$.

7. Show that the right-circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base.

8. Show that of all the rectangles of given area, the square has the smallest perimeter.

64 If the tangent to the curve $y = x^3 + ax + b$, at $P(1, -6)$ is parallel to the line $y - x = 5$, find the values of $and\ b$.

ANS: $y = x^3 + ax + b$, $\frac{dy}{dx} = 3x^2 + a$

$\frac{dy}{dx}\ at\ P(1, -6) = 3 + a$,

$y - x = 5$, $\frac{dy}{dx} = 1$, $3 + a = 1 \Rightarrow a = -2$

$(1, -6) lie$ on the curve $y = x^3 + ax + b$

$a + b = -7$ solve , $b = -5$

65 Find the local maximum and local minimum values of the function

$f(x) = sinx + \frac{1}{2}cos2x,\ \ 0 < x < \frac{\pi}{2}$

$f(x) = sinx + \frac{1}{2}cos2x,\ \ 0 < x < \frac{\pi}{2}$

$f'(x) = cosx + \frac{1}{2}(-2sin2x) = cosx - sin2x$

$f'(x) = 0\ \ then$

$cosx - sin2x = 0$

$cosx(1-2sinx)=0$, $cosx=0, sinx=\frac{1}{2}$ $x=\frac{\pi}{2},\frac{\pi}{6}$

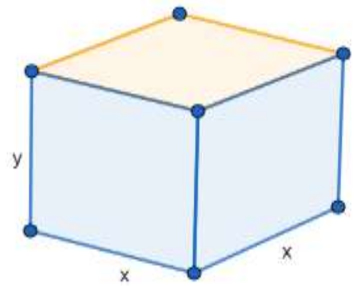
$f''(x) = -sinx - 2\,cos2x$

$f''\left(\frac{\pi}{6}\right) = -\frac{3}{2} < 0$

Local max. value = $\frac{3}{4}$ at $\frac{\pi}{6}$

$f''\left(\frac{\pi}{2}\right) = 1 > 0$, Local min. value =1/2 at $\frac{\pi}{2}$

66 An open rectangular tank , with a square base and vertical sides, is to be constructed of metal sheet to hold a given quantity of water as shown below:



Based on the above information answer the following

i) If x represents the side of the square base and y represents the depth of the tank, then the volume V of the tank in terms of x and y is _______

a) $V = x^2y$ b) $V = xy$ c) $V = x^2y^2$ d) xy^2

ii) Let S be the area of the metal sheet required to construct the tank, then

a) $S = 2x^2 + xy$ b) $S = x^2 + xy$ c) $S = 5x^2$ d) $S = x^2 + 4xy$

iii) The area of the metal sheet S expressed as a function of x is _______

a) $S = x^2 + \frac{x}{4V}$ b) $S = 2x^2 + \frac{4V}{x}$ c) $S = x^2 + \frac{4V}{x}$ d) $S = 2x^2 + \frac{4V}{x^2}$

iv) The area of the metal sheet S will be minimum when $x =$ ______

a) $V^{\frac{1}{3}}$ b) $V^{\frac{2}{3}}$ c) $(2V)^{\frac{1}{3}}$ d) $2V$

v) Cost of the metal will be least when the depth (y) is _________.

a) x b) $\frac{x}{2}$ c) x d) $2x$

ANS: i) a) $V = x^2y$ ii) d) $S = x^2 + 4xy$

iii) c) $S = x^2 + \frac{4V}{x}$ iv) c) $(2V)^{\frac{1}{3}}$ v) b) $\frac{x}{2}$

67 A closed right circular cylinder has volume 2156 cubic units. What should be the radius of the base so that its total surface area may be minimum?

ANS: V = $\pi\, r^2h$

S = $2\pi rh + 2\,\pi\, r^2$

$\Rightarrow S = 2\pi r\left(\frac{2156}{\pi\, r^2}\right) + \pi\, r^2$

S = $\frac{4312}{r} + \frac{44}{7}r^2$

$\frac{dS}{dr} = 0 \Rightarrow r = 7$

$\frac{d^2s}{dr^2} > 0$

68 A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter P of the window is 10 metres. We have to find the dimensions of the window so as to admit maximum light through the whole opening.

Based on the above information answer the following:

i) Let x be side of a rectangle and r be the radius of semicircle. Then Perimeter P =______

a) $4x + 2r + \pi r$ b) $2x + 2r + \pi r$ c) $2x + 4r + \pi r$ d) $2x + r + 2\pi r$

ii) To admit maximum possible light, area of window should be maximum. Let A be the area of the window, then

a) $A = \pi r^2 + 2xr$ b) $A = \frac{1}{2}\pi r^2 + 4xr$ c) $A = \frac{1}{2}\pi r^2 + 2xr$ d) $A = \frac{1}{2}\pi r^2 + xr$

iii) If $\frac{dA}{dr} = 0 \Rightarrow r =$ _________

a) $\frac{10}{4+\pi}$ b) $\frac{5}{4+\pi}$ c) $\frac{20}{4+\pi}$ d) $\frac{10}{4-\pi}$

iv) Dimensions of the rectangle are ______ and ________.

a) $\frac{10}{4+\pi}, \frac{5}{4+\pi}$ b) $\frac{20}{4+\pi}, \frac{5}{4+\pi}$ c) $\frac{20}{4-\pi}, \frac{5}{4-\pi}$ d) $\frac{10}{4+\pi}, \frac{20}{4+\pi}$

v) $\frac{d^2A}{dr^2} =$ ________

a) $(4+\pi)$ b) $-(4+\pi)$ c) $(4-\pi)$ d) $-(4-\pi)$

ANS: i) b) $2x + 2r + \pi r$ ii) c) $A = \frac{1}{2}\pi r^2 + 2xr$

iii) a) $\frac{10}{4+\pi}$ iv) d) $\frac{10}{4+\pi}, \frac{20}{4+\pi}$ v) b) $-(4+\pi)$

ALPHA 12